ON TOTAL ABSOLUTE CURVATURE

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C E R T I F I C A T E

This is to certify that the contents of this thesis entitled "ON TOTAL ABSOLUTE CURVATURE" *is the original research work of Mr. Viqar Azam Khan carried out under my supervision. He has fulfilled the prescribed conditions given in the ordinances and regulations of Aligarh Muslim University, Aligarh.*

I further certify that the work has not been submitted either partly or fully to any other university or institution for the award of any degree.

(S. IZHAR HUSAIN)

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VKhan

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P R E F A C E

The present thesis is mainly devoted to the study of Immersions and total absolute curvature. Immersion has been the subject of interest since the time of E. Cartan. However Whitney's theorem on immersion was the beginning of the serious study of the subject. In order to study the geometry of a manifold, it was found to be a lot more convenient to first embeded it into a manifold of which the geometry is known and then study the geomety of the underlying manifold vis-a-vis the known ambient manifold. Of these geometric properties, one of the prime importance is the "Total absolute curvature" of the immersed manifolds which being invariant depending on the way the manifold is immersed in the ambient space, and hence forms the main theme of the thesis.

The thesis comprises five chapters. The first chapter is introductory and is basically intended to make the thesis as self-contained as possible. For the convenience of ready references and to fix up the terminology some known and relevant results on Geometry of submanifolds, Homology, Morse theory and total absolute curvature, have been collected in this chapter.

The second chapter is devoted to a problem of immersion. A. Weinstein [37], in his paper "Positivity curved compact manifolds in R^{n+2}" has proved that a 4-dimensional positively curved, compact manifold in R^6 is a real homology sphere. He also raised the problem of classifying such manifolds M^n of arbitrary dimension n isometrically immersed in R^{n+2}. In this chapter we answer the problem raised by him. Infact we have proved that a compact positively curved manifold M^n isometrically immersed in R^{n+2} is a real homology sphere. Our technique is entirely different from that of A. Weinstein and it mainly depends on critical point theory of the height function and the observation that the mean curvature vector, in this situation is a globally defined vector field normal to the manifold M^n.

When a compact C^∞-manifold is immersed in another manifold, usually Euclidean space, this gives rise to various curvature measures. By integeration it is possible to associate with immersion a real number called the total absolute curvature. If we consider the infemum of these numbers over all immersions, then the resulting number depends only on the given manifold that is it becomes an invariant of the manifold. An immersion for which the total absolute curvature attains infemum is called tight immersion. A natural problem is to consider for what manifolds, tight immersions are possible and what are geometrical properties of manifolds admitting tight immersion. C.S. Chen [7] has shown that if a non-negatively curved compact manifold M^n of dimension n admits a tight isometric immersion in R^{n+2}, then it is diffeomorphic to the product of spheres. In Chapters 3 & 4, we have generalized the results of C.S. Chen for arbitrary codimension with the restriction that the normal bundle is flat.

We have obtained estimate of the infemum of the non-degenerate critical points over all immersions in Chapter 3. We have shown that under these condition $\mu(M) \leq 2^N$, where N is the codimension of the manifold M in the euclidean space R^{n+N}.

The above inequality becomes sharp when M has strictly positive sectional curvature at some point. Hence this explains if $\mu(M) = 2^N$, then $T(X,X)$, the shape operator occupies the whole of the first region of R^N provided no X is asymptotic. Therefore under these additional conditions, we succeed in splitting the tangent space. In Chapter 4, we prove that the leaves so obtained are parallel and totally geodesic. This conclusion ultimately helps us in proving that M is diffeomorphic to the product of spheres.

Deviating a little from the problem of immersion, we prove similar kind of result in a different perspective in Chapter 5. Infact we establish a classification theorem for the Bochner flat Kaehler manifolds when the curvature tensor commutes with the Ricci operator. We conclude that such manifolds are either flat or complex space forms or locally the Riemannian product of two complex space forms one of constant curvature C and other -C.

The thesis ends with a bibliography which by no means is an exhaustive one but contains only those references which are referred to in the text.

ABRIVIATIONS

(m.n.p) : The number in three slot denotes a mathematical relation and means p-th relation occuring in the n-th section of Chapter m. Similar numbering is adopted for definitions, propositions and lemma etc.

$T(M)$: Tangent bundle of the manifold M

$\nu(M)$: Normal bundle of the manifold M

$\mu(M)$: Infemum of all non-degenerate critical points over all the morse function.

$\bar{\mu}(M)$: Infemum of all non-degenerate critical point with index $\neq o,n$ over all morse functions where n is the dimension of M.

$T(X,Y)$: Shape operator

S_e : Weingarten map corresponding to the normal vector field e.

$K(p)$: Sectional curvature of the plane section p of the tangent space.

$H(X)$: Holomorphic sectional curvature

$Q(X,Y)$: Ricci operator

$\tau(M,F)$: Total absolute curvature of M with respect to the immersion $F : M^n \longrightarrow R^{n+N}$.

$|S_o^{n+N-1}(1)|$: Volume of the unit (n+N-1)-sphere centred at origin in R^{n+N}.

$H^n(X,R)$: n-th singular cohomology group with coefficients in R.

C O N T E N T S

CHAPTER I

I N T R O D U C T I O N

The purpose of this chapter is to give a brief resume of the results in the geometry of submanifolds, Homology, Morse theory and Total absolute curvature. Muchthough all these results are readily available in review articles and some even in standard books e.g. K. Nomizu and S. Kobayashi [17], S.T. Hu [19], J. Millnor [25], T.E. Cecil and P.J. Ryan [30] and many others, nevertheless we have collected them here to fix up our terminology. Only those definitions and results have been given which are relevant in the subsequent chapters.

1. GEOMETRY OF SUBMANIFOLDS:

To study the geometry of submanifolds, sometimes, it becomes more convenient to first embed it into a manifold of which the geometry is known and then study the geometry which is induced on it.

Let M and $\bar{M}$ be n and $(n+N)$-dimensional Riemannian manifolds. M is said to be a Submanifold of $\bar{M}$, if there exists an immersion $F : M \longrightarrow \bar{M}$. For the submanifold M of $\bar{M}$, for convenience we denote $F(x) \in \bar{M}$ by the same letter x.

Let $T(M)$ and $T(\bar{M})$ denote the tangent bundle of M and $\bar{M}$ respectively. F is said to be an <u>isometric immersion</u> if the differential map $F_* : T(M) \longrightarrow T(\bar{M})$ preserves the metric i.e. for $X, Y \in T(M)$

$$g(F_*X, F_*Y) = g(X,Y) \tag{1.1.1}$$

where we denote by the same letter g the Riemannian metric on M and $\bar{M}$ respectively. For local calculations, we identify $T(M)$ and $F_*(T(M))$ through this isomorphism. Hence a tangent vector in $T(\bar{M})$ tangent to M shall mean a tangent vector which is the image of an element in $T(M)$ under F_*. Those tangent vectors of $T(\bar{M})$ which are normal to M form the normal bundle $\nu(M)$. The Riemannian connexion $\bar{\nabla}$ on $\bar{M}$ induces Riemannian connexions ∇ and $\nabla^{\perp}$ in the tangent bundle of M and the normal bundle $\nu(M)$ respectively and they are related by <u>Gauss and Weingarten</u> formulae.

$$\bar{\nabla}_X Y = \nabla_X Y + T(X,Y) \tag{1.1.2}$$

$$\bar{\nabla}_X N = -S_N X + \nabla^{\perp}_X N. \tag{1.1.3}$$

For $X,Y \in T(M)$ and $N \in \nu(M)$, $\nabla_X Y$ (resp. $-S_N X$) and $T(X,Y)$ (resp. $\overset{\perp}{\nabla}_X N$) are tangential and normal component of $\bar{\nabla}_X Y$ (resp. $\bar{\nabla}_X N$) where $T(X,Y)$ is a normal valued symmetric tensor called the Shape operator, $S_N X$ is a symmetric (1,1) tensor on M and they are related by

(1.1.4) $\qquad g(S_N X,Y) = g(T(X,Y),N)$

If we denote the curvature tensor corresponding to $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ by $\bar{R}$, R and $\overset{\perp}{R}$ respectively, then the Gauss, Coddazi and Ricci Equations are given by [17]

(1.1.5) $\qquad \bar{R}(X,Y;Z,W) = R(X,Y;Z,W) + g(T(X,Z),T(X,W)) - g(T(Y,Z),T(X,W))$

(1.1.6) $\qquad [\bar{R}(X,Y)Z]^{\perp} = (\bar{\nabla}_X T)(Y,Z)-(\bar{\nabla}_Y T)(X,Z)$

(1.1.7) $\qquad \bar{R}(X,Y;N_1,N_2) = \overset{\perp}{R}(X,Y;N_1,N_2)-g([S_{N_1},S_{N_2}]X,Y).$

Where in (1.1.6) $[\bar{R}]^{\perp}$ denotes the normal component of $[\bar{R}]$ and

(1.1.8) $\qquad (\bar{\nabla}_X T)(Y,Z) = \bar{\nabla}_X T(Y,Z)-T(\bar{\nabla}_X Y,Z)-T(Y,\bar{\nabla}_X Z)$

For each plane p in the tangent space $T_x(M)$, the Sectional Curvature $K(p)$ for p is defined by, (1.1.9) $K(p) = R(X_1,X_2;X_2,X_1) = g(R(X_1,X_2)X_2,X_1)$.

Where $\{X_1,X_2\}$ is the orthonormal basis for p. This can be seen that K(p) is independent of the choice of an orthonormal basis $\{X_1,X_2\}$. If K(p) is constant for all planes p in $T_x(M)$ and for all points $x \in M$, then M is called a <u>Space of constant curvature</u>. The following theorem is due to F. Schur [17].

<u>THEOREM</u> (1.1.1): Let M be a connected Riemannian manifold of dimension ≥ 3. If the sectional curvature K(p), where p is a plane in $T_x(M)$, depends only on x, then M is a space of constant curvature.

<u>COROLLARY</u> (1.1.1): For a space of constant curvature k, we have,

(1.1.10) $R(X,Y)Z = k(g(Z,Y)X - g(Z,X)Y)$

From (1.1.5) it follows that the sectional curvature $\bar{K}$ and K of $\bar{M}$ and M of plane section spanned by vectors X and Y satisfy

(1.1.11) $\bar{K}(X,Y) = K(X,Y)+g(T(X,Y),T(X,Y))-g(T(X,X),T(Y,Y))$.

Since S_N is symmetric, there exists an orthonormal local frame $(e_1,\ldots,e_n)$ on M such that $S_N e_i = \lambda_i e_i$ where λ_i depends on M. Now let us consider the local formalism. For this we choose a local field of orthonormal frame $e_1, e_2, \ldots, e_{n+N}$ in $\bar{M}$ such that restricted to M, the vectors $e_1, e_2, \ldots, e_n$ are tangential to M and the rest of the N vectors $e_{n+1}, \ldots, e_{n+N}$ are normal to M. We shall make use of the following convention of the ranges of indices

$$1 \leq A, B, C, \ldots, \leq n+N$$

$$1 \leq i, j, k, \ldots, \leq n$$

$$n+1 \leq r, s, t, \ldots, \leq n+N.$$

The structure equation of $\bar{M}$ are given by

(1.1.12) $$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0$$

(1.1.13) $$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}$$

where $\Omega_{AB} = \frac{1}{2} \sum_{C,D} \bar{R}_{ABCD}\, \omega_C \wedge \omega_D$

where $\omega_1, \ldots, \omega_{n+N}$ is the dual frame to $e_1, \ldots, e_{n+N}$ and ω_{AB} are the connexion forms. If we restrict these forms to M, $\omega_r = 0$ and (1.1.12) gives

$$0 = d\omega_r = -\sum_j \omega_{rj} \wedge \omega_j.$$

Using Cartan's lemma, we write

(1.1.14) $\qquad \omega_{ri} = \sum_r A^r_{ij}\, \omega_j, \qquad\qquad A^r_{ij} = A^r_{ji}.$

From these formulas, we obtain

(1.1.15) $\qquad d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j\,, \qquad\qquad \omega_{ij} + \omega_{ji} = 0$

(1.1.16) $\qquad d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}$

where $\quad \Omega_{ij} = \frac{1}{2}\, R_{ijkl}\, \omega_k \wedge \omega_l.$

2. <u>HOMOLOGY</u>:

Let R^n be an n-dimensional Euclidean space consider the following set

$$\Delta_n = \{(x_1,\ldots,x_n) \in R^n / \sum_{i=1}^{n} x_i \le 1, \quad x_i \ge 0\}$$

Clearly Δ_n, is a compact subset of R^n, is called an n-simplex. $(0,0,\ldots,0) \in \Delta_n$ is called the 0-th vertex and the points $v_i = (0,\ldots,0,1,0,\ldots,0)$ with 1 at the i-th place is called the i-th vertex. The closed subspace

$$\Delta_n^{(o)} = \{(x_1,x_2,\ldots,x_n) \in \Delta_n / \sum_{i=1}^{n} x_i = 1\}$$

is called the <u>0-th face of Δ_n</u> and

$$\Delta_n^{(i)} = \{(x_1, x_2, \ldots, x_n) \in \Delta_n / x_i = 0\}$$

is called <u>i-th face of Δ_n</u>. Clearly $v_i \notin \Delta_n^{(i)}$. For completeness, we define $\Delta_o = \{0\}$. Assume $n > o$ and define the map

$$k_i : \Delta_{n-1} \longrightarrow \Delta_n \quad , \quad (i = o, 1, \ldots, n)$$

by $k_o(x_1, x_2, \ldots, x_{n-1}) = (1 - \sum_{i=1}^{n-1} x_i, x_1, \ldots, x_{n-1})$

and $k_i(x_1, x_2, \ldots, x_{n-1}) = (x_1, \ldots x_{i-1}, o, x_i \ldots, x_{n-1})$.

It is easy to verify that

$k_i : \Delta_{n-1} \longrightarrow \Delta_n$ is an embedding and

$$k_i(\Delta_{n-1}) = \Delta_n^{(i)} .$$

Next assume $n > 1$ and consider the map

$$\Delta_{n-2} \xrightarrow{k_i} \Delta_{n-1} \xrightarrow{k_i} \Delta_n$$

Then one can prove, $k_i o\, k_j = k_j \, o\, k_{i-1}$.

<u>DEFINITION</u> (1.2.1): Let X be a topological space. A continuous map

$$\sigma : \Delta_n \longrightarrow X$$

is called a <u>Singular p-simplex</u>.

Since Δ is fixed for each p, the function σ carries the same information as does the set $\sigma(\Delta_p)$

Let $S_n(X)$ be the set of all n-Singular Simplexes. Clearly for $m \neq n$,

$$S_m(X) \cap S_n(X) = \phi$$

Assume $n > 1$ and let $\sigma \in S_n(X)$. For $i = o,1,\ldots,n$ Consider the composition

$$\sigma \circ k_i : \Delta_{n-1} \longrightarrow X$$

Clearly $\sigma \circ k_i \in S_{n-1}(X)$. We put $\sigma \circ k_i = \sigma^{(i)}$ which is called the <u>i-th face of σ</u>. We have clearly,

$$[\sigma^{(i)}]^{(j)} = [\sigma^{(j)}]^{(i-1)}, \quad \text{for} \quad n > 1$$

for all non negative integers n, let $C_n(X)$ be the free abelian group generated by $S_n(X)$. Now we define

$$\partial : C_n(X) \longrightarrow C_{n-1}(X)$$

by $$\partial(\sigma) = \sum_{i=1}^{n} (-1)^i \sigma^{(i)} .$$

For completeness, we define

$$C_n(X) = 0, \quad n < 0 .$$

Then ∂ defines a homomorphism, and

$$\partial \circ \partial = 0$$

Thus we obtain a semi exact sequence

$$\cdots\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \cdots\cdots\cdots$$

The elements $\gamma \in C_n(X)$ are called <u>Singular Chain</u> and $\partial\gamma \in C_n(X)$ is called the <u>boundary of chain γ</u>. Define

$$Z_n(X) = \{\text{Ker } \partial/\partial : C_n(X) \longrightarrow C_{n-1}(X)\}$$

and

$$B_n(X) = \{\text{Im } \partial/\partial : C_{n+1}(X) \longrightarrow C_n(X)\}$$

The elements of Z_n are called <u>Cycles</u> and that of B_n are called <u>boundaries</u>. We define

$$H_n(X) = {}^{Z_n(X)}/B_n(X)$$

<u>DEFINITION</u> (1.2.2): $H_n(X)$ is called the n-dimensional <u>Singular homology group</u>.

<u>DEFINITION</u> (1.2.3): Let $C^n(X,R) = \text{Hom }(C_n(X),R)$.

The elements of $C^n(X,R)$ are called <u>real singular cochains</u>.

For $n < o$, we have

$C^n(X,R) = 0$, as $C_n(X) = 0$.

For every n, define homomorphism

$$\delta : C^n(X,R) \longrightarrow C^{n+1}(X,R)$$

as follows:

If $n < 0$, $C^n(X,R) = 0$ and $\delta = 0$.

Assume $n \geq 0$, let $\phi \in C^n(X,R)$

$$\delta(\phi) = \phi o \partial : C_{n+1}(X) \longrightarrow R$$

Clearly δ is a homomorphism. We call δ the <u>coboundary operator</u>.

<u>LEMMA</u> (1.2.1): $\delta \circ \delta = 0$.

As in the previous case, we define $Z^n(X,R)$ to be the Ker δ and $B^n(X,R)$ as the image of δ and call them <u>cocycles</u> and <u>coboundaries</u> respectively.

<u>DEFINITION</u> (1.2.4): $H^n(X,R) = Z^n(X,R)/_{B^n(X,R)}$

is called the n-dimensional real <u>Singular Cohomology group</u>.

<u>COHOMOLOGY OF FORMS</u>: Let M be an n-dimensional differentiable manifold. Then we know the following algebra of differential forms

$$C(M) = \sum_{k=0}^{n} C^k(M).$$

Where $C^k(M)$ is the set of differential k-forms and is associated with M, called the Exterior algebra of differential forms. There is a unique exterior differential operator

$$d : C(M) \longrightarrow C(M)$$

which maps a k-form into a (k+1)-form and satisfies $d \circ d = 0$.

Also, $C^k(M) = 0$, if $k > n$ and $k < o$

and hence we have the following semi exact sequence:

$$\cdots\cdots \xrightarrow{d} C^{k-1}(M) \xrightarrow{d} C^k(M) \xrightarrow{d} C^{k+1}(M) \xrightarrow{d} \cdots\cdots\cdots$$

let $Z^k(M) = \{\eta \in C^k(M)/d\eta = 0\}$

$$= \{\text{Ker } d/d : C^k(M) \longrightarrow C^{k+1}(M)\}.$$

i.e. set of all closed forms. Define

$$B^k(M) = d(C^{k-1}(M))$$

$$= \{ d\eta : \eta \in C^{k-1}(M)\}$$

Now as before we have the following lemma

<u>LEMMA</u> (1.2.2): For each k, $B^k(M) \subseteq Z^k(M)$. Infact it is a linear subspace.

DEFINITION (1.2.5): $H^k(M) = Z^k(M)/_{B^k(M)}$ called the k-dimensional De-Rham cohomology group of differentiable manifold M.

REMARK : i) $H^o(X) = Z^o(X)/_{B^o(X)} = Z^o(X)$

ii) $H^n(X) = C^n(X)/_{B^n(X)}$

iii) $H^{n+i}(X) = 0$.

$H^k(M)$ consist of equivalence classes of the type $[\eta]$, where η is a k-form which is closed and not exact. Also we observe that

$$\xi \in [\eta]$$

$$\text{if} \quad \xi = \eta + d\alpha \quad \text{for some } (k-1)\text{-form } \alpha$$

i.e. $\xi - \eta$ is cohomologus to zero.

From Poincare's lemma it is known that every closed form in R^n is exact. Therefore we have,

LEMMA (1.2.3): $H^k(R^n) = 0$

LEMMA (1.2.4): If $[\xi] \in H^k(M)$ and $[\eta] \in H^l(M)$, then $[\xi \wedge \eta] \in H^{k+l}(M)$.

LEMMA (1.2.5): For sphere S^n, we have,

$$H^o(S^n) = H^n(S^n) \approx R \quad \text{and } H^p(S^n) = 0, \quad 1 \leq p \leq n-1$$

Now we can state the famous De-Rham's theorem.

THEOREM (1.2.1): For every integer k, the De-Rham's Cohomology group $H^k(M)$ is isomorphic to $H^k(M,R)$.

DEFINITION (1.2.6): $\dim H^k(M)$ is defined to be $b_k(M)$, the k-th betti number of the manifold, and

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \, b_k(M)$$

is called the Euler Characteristic

3. MORSE THEORY:

The underlying idea of Morse theory is that a considerable knowledge of the topological properties of a manifold can be obtained by studying the number and the nature of the critical points of a real valued function defined over M.

DEFINITION (1.3.1): A smooth function ϕ defined over a manifold M has a critical point at x if $d\phi$ vanishes at x. In terms of local coordinates (x^i) this means that at x,

$$\left(\frac{\partial \phi}{\partial x^i}\right) = 0, \quad \text{for all } i$$

DEFINITION (1.3.2): Let $\phi : M \longrightarrow R$ be a differentiable function. The hessian of ϕ at $x \in M$ is an $(n \times n)$-matrix

$\left(\left(\frac{\partial^2\phi}{\partial x^i \partial x^j}\right)\right)$ where $(x^1, x^2, \ldots, x^n)$ are local coordinates at x.

DEFINITION (1.3.3): Let $\phi : M \longrightarrow R$ be a differentiable function. A critical point x of ϕ is said to be non-degenerate if the hessain of ϕ at x is non singular otherwise the critical point is said to be degenerate.

REMARK : The degeneracy or non-degeneracy is independent of the local coordinates system around the critical point.

DEFINITION (1.3.4): Let $x \in M$ be the non-degenerate critical point of the function $\phi : M \longrightarrow R$. Then the maximal dimension of the subspace of $T_x(M)$ on which the hessain of ϕ is negative definite is called the index of x.

The behaviour of ϕ in a neighbourhood of a non-degenerate critical point is determined by the index as follows:

LEMMA (1.3.1): Let $x \in M$ be a non-degenerate critical point of $\phi : M \longrightarrow R$ of index k. Then there is a local coordinate system $x^1, x^2, \ldots, x^n$ in a neighbourhood U with origin at x such that the identity

$$\phi = \phi(x) - (x^1)^2 - (x^2)^2 - \ldots - (x^k)^2 + (x^{k+1})^2 + \ldots + (x^n)^2$$

holds throughout U.

LEMMA (1.3.2): The non-degenerate critical point of a function are isolated. Thus a function with compact domain has only finitely many non-degenerate critical points provided it has no degenerate critical point.

DEFINITION (1.3.5): A function ϕ which has only non-degenerate critical points is called a non-degenerate function or a Morse function.

Let $\mu_k(\phi,M)$ be the number of critical points of ϕ of index k. Let

$$\mu(\phi,M) = \sum_{k=o}^{n} \mu_k(\phi,M)\ ,\ \bar{\mu}(\phi,M) = \sum_{k=1}^{n-1} \mu_k(\phi,M)$$

$$\mu_k(M) = \text{Inf}\ \{\ \mu_k(\phi,M)/\phi : M \longrightarrow R\}$$

and $\mu(M) = \text{Inf}\ \{\ \mu(\phi,M)/\phi : M \longrightarrow R\}$, $\bar{\mu}(M)=\text{Inf}\{\bar{\mu}(\phi,M)/\phi:M \longrightarrow R\}$.

where infemum is taken over all the non-degenerate functions on M.

The Morse inequalities which hold for a non-degenerate function ϕ on a compact manifold M give

$$\mu_k(\phi,M) \geq \mu_k(M) \geq b_k(M,F).$$

Also $\quad \mu(\phi,M) \geq \mu(M) \geq \sum_i \mu_i(M).$

4. TOTAL ABSOLUTE CURVATURE:

The theory of total absolute curvature can be considered to have been originated in 1929 with a paper of W. Fenchel [12]. In that paper he proved that for a closed space curve C in R^3 of class $\geq C^3$

$$\frac{1}{\pi} \int_C |k(S)| \, ds \geq 2. \tag{1.4.1}$$

Where $k(S)$ is the ordinary curvature of the curve C. Equality holds if and only if C is a plane convex curve. This idea was further generalized by K. Borsuk for a curve in R^n, $n \geq 3$.

Now before coming to the general case i.e. the total absolute curvature of a compact manifold in Euclidean space, let us consider the Gauss map of M^n immersed in R^{n+N}. In the classical theory of surfaces Σ in R^3, the Gauss map is defined by

$$\nu : \Sigma \longrightarrow S_o^2$$

Such that $\nu(p) = P^*$

Where the unit normal to the surface Σ at P is parallel to the position vector P^* of the sphere S_o^2 (i.e. the unit vector originating from origin and with end point as P^*)

The idea of Gauss map and Gaussian curvature is generalized to differentiably immersed submanifolds of Euclidean spaces by S.S. Chern [9] and thereby gave the notion of total absolute curvature. Let

$$F : M \longrightarrow R^{n+N}$$

be an immersion of an n-dimensional C^{∞}-manifold into (n+N)-dimensional Euclidean space. The immersion F gives rise to a unit normal bundle $\gamma(M)$ over M whose bundle space consists of all the pairs (p,e) where $p \in M$ and e is the unit normal vector at $F(p)$. If S_0^{n+N-1} is the unit sphere in R^{n+N} centred at origin, we define the Gauss map on $\nu(M)$ as follows:

$$\nu : \nu(M) \longrightarrow S_0^{n+N-1}$$

Such that $\nu(p,e) = e_0$

that is under Gauss map every unit normal vector at $F(p)$ is mapped to the end point of a unit vector through origin parallel to it.

If dv is the volume element of M and $d\sigma$ be (N-1)-differential form on $\nu(M)$ such that its restriction to each fibre is the volume element of $\nu(M)$. Let $d\Sigma$ be the volume element of S_o^{n+N-1}, then we define $G(p,e)$ on $\nu(M)$ by

$$(1.4.2) \qquad \nu^* \, d\Sigma = G(p,e) \, dv \wedge d\sigma.$$

Where ν^* is the dual mapping on differential forms induced by ν. $G(p,e)$ is called the <u>Lipschitz Killing Curvature</u>.

<u>DEFINITION</u> (1.4.1): The total absolute curvature $K^*(p)$ at $p \in M$ of the immersion $F : M^n \longrightarrow R^{n+N}$ is defined by

$$(1.4.3) \qquad K^*(p) = \frac{1}{|S(1)^{n+N-1}|} \int_{c \in \nu(M)} |G(p,e)| \, d\sigma.$$

Where $|S(1)^{n+N-1}|$ denotes the volume of $(n+N-1)$-dimensional unit sphere. The total absolute curvature of M is defined by

$$(1.4.4) \qquad \tau(M,F) = \int_M K^*(p) \, dv.$$

In the case of a 2-dimensional (orientable) closed surfaces M^2 immersed in R^3, the Lipschitz curvature becomes the Gaussian curvature K and (1.4.4) becomes

$$\tau(M^2,F) = \frac{1}{2\pi} \int_M |K| ds$$

The total absolute curvature of the immersion remains unchanged if $F(M^n)$ is considered as a submanifold of a higher dimensional Euclidean space which contains R^{n+N} as a linear subspace.

Following are the main results on total curvature by S.S. Chern and R.K. Lashoff [9].

<u>THEOREM</u> (1.4.1): Let M^n be a compact, oriented C^∞-manifold immersed in R^{n+N}. Its total curvature satisfies the inequality

$$\frac{1}{|S(1)^{n+N-1}|} \int_M K^*(p)\, dv \geq 2.$$

<u>THEOREM</u> (1.4.2): Under the hypothesis of the theorem (1.4.1), if

$$\frac{1}{|S(1)^{n+N-1}|} \int_M K^*(p)\, dv < 3$$

then M^n is homeomorphic to a sphere of n-dimension.

<u>THEOREM</u> (1.4.3): Under the same hypothesis, if

$$\frac{1}{|S(1)^{n+N-1}|} \int_M K^*(p)\, dv = 2,$$

then M^n belongs to a linear subvariety R^{n+1} of dimension n+1, and is imbedded as a convex hypersurface in R^{n+1}. The converse of this is also true.

CHAPTER II

POSITIVELY CURVED n-MANIFOLDS IN R^{n+2}

In an attempt to classify positively curved compact Riemannian manifolds M^n, A Weinstein [37], under the additional assumption that M^n are isometrically immersed in R^{n+2}, proved that Pontryagin and Stiefle whitney classes of these manifolds are trivial as well as their second betti number is zero. In the case of 4-dimensional, positively curved compact M^4 isometrically immersed in R^6, he has shown that they are real homology spheres. In this Chapter we study positively Curved Compact Riemannian manifold M^n isometrically immersed in R^{n+2}, we prove that they are real homology spheres. Our technique depends on the critical point theory of the height function and the observation that in such cases the mean curvature vector is nonzero throughout the manifold and thus defines a global unit normal vector field which serves the purpose.

1. PRELIMINARIES:

Let M be an n-dimensional Riemannian manifold which is isometrically immersed in an (n+N)-dimensional Riemannian manifold $\bar{M}$. The relationship between the curvature tensor $\bar{R}$ of $\bar{M}$ and the curvature tensor R of M is given by the Gauss Equation

$$(2.1.1) \qquad \bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(T(Y,Z),T(X,W)) - g(T(Y,W),T(X,Z))$$

where X,Y,Z,W are arbitrary tangent vectors to M and T is the second fundamental form of the immersion of M into $\bar{M}$.

If $e_1, e_2, \ldots, e_n$ is the orthonormal basis of $T_x(M)$, then the Ricci tensor Q of M is given by

$$(2.1.2) \qquad Q(X,Y) = \sum_{i=1}^{n} g(R(e_i,X)Y,e_i); \; X,Y \in T_x(M)$$

The mean curvature vector is defined as

$$(2.1.3) \qquad H = \frac{1}{n} \left(\sum_{r=n+1}^{n+N} \left(\sum_{i=1}^{n} T^r(e_i,e_i) \, e_r \right) \right)$$

where $T^r(e_i,e_j)$ are the components of $T(e_i,e_j)$.

The sectional curvature of the plane section spanned by the orthonormal unit vectors e_i and e_j is defined by

$$(2.1.4) \qquad K_{ij} = R(e_i,e_j;e_j,e_i)$$

A smooth function ϕ defined over a manifold M has a critical point at x if $d\phi$ vanishes at x. If x is a critical point of ϕ, one examines the hessian of ϕ which is represented in local coordinates by the symmetric matrix

$$H_x = \left(\left(\frac{\partial^2\phi}{\partial x^i \partial x^j}\right)\right)$$

Let $F : M^n \longrightarrow R^m$ be the smooth immersion and $e \in S^{m-1}$ the unit hypershere centred at origin in R^m. The real valued function h_e on M defined by

$$h_e(x) = \langle F(x), e\rangle$$

is known as the height function. Where $\langle , \rangle$ is the Euclidean innerproduct of R^m and F is interprated as a function with values in the vector space R^m. We have the following results on the critical point theory of the height of function [30].

<u>LEMMA</u> (1.1.1): Let $F : M^n \longrightarrow R^m$ be an immersion and let $e \in S^{m-1}$, then

i) h_e has a critical point at $x \in M$ if, and only if, e is orthogonal to $T_x(M)$ ie, $(x,e) \in \nu(M)$, the normal bundle of M.

ii) Suppose h_e has a critical point at x. Then for $X,Y \in T_x(M)$, the hessian H_x at x satisfies

(2.1.5) $$H_x(X,Y) = \langle S_eX,Y\rangle$$

where S_e is the Weingarten map defined by

(2.1.6) $$\langle S_eX,Y\rangle = \langle T(X,Y),e\rangle$$

T, being the shape operator of the immersion F.

<u>DEFINITION</u> (2.1.1): Let x be a critical point of the function $\phi : M \longrightarrow R$

i) x is called a <u>degenerate critical point</u> if rank $(H_x) < n$, where n is the dimension of M.

ii) If rank $(H_x) = n$, x is called a <u>non degenerate critical point</u> or in other words x is non degenerate if and only if the matrix $\left(\left(\frac{\partial^2\phi}{\partial x^i \partial x^j}\right)\right)$ is non singular.

<u>LEMMA</u> (2.1.2): Let $e \in T_x^{\perp}(M)$

i) h_e has a degenerate critical point if, and only if, S_e is singular.

ii) If h_e has a non-degenerate critical point at x, then the index of h_e at x is equal to the number of negative eigen values of S_e.

2. POSITIVELY CURVED M^n INTO R^{n+2}

Our main aim in this section is to establish the following theorem:

THEOREM (2.2.1): Let M be an n-dimensional positively curved compact Riemannian manifold isometrically immersed in R^{n+2}. Then M is a real homology sphere.

We first establish the following:

LEMMA (2.2.1): Let M be an n-dimensional Riemannian manifold of strictly positive sectional curvature, isometrically immersed in R^{n+2}. Then the mean curvature vector is a non vanishing glob-ally defined vectorfield.

PROOF: The curvature tensor R of M is given by

$$(2.2.1) \qquad R(X,Y;Z,W) = g(T(Y,Z),T(X,W))-g(T(X,Z),T(Y,W))$$

for $X,Y,Z \in \mathfrak{X}(M)$

Since $\bar{R}(X,Y;Z,W) = 0$ as R^{n+2} is flat. By [37] R is (positive) definite.

The Ricci curvature of M is given by

$$(2.2.2) \qquad Q(X,Y) = ng(T(X,Y),H) - \sum_{i=1}^{n} g(T(X,e_i),T(Y,e_i))$$

where $e_1, e_2, \ldots, e_n$ is a local frame on M and

$$H = \frac{1}{n} \sum_{r=n+1}^{n+2} \Big(\sum_{i=1}^{n} T^r(e_i, e_i) \Big) e_r$$

is the mean curvature vector. We observe that

$$Q(e_j, e_j) = \sum_{i=1}^{n} K_{ij}$$

where $K_{ij} = R(e_i, e_j; e_j, e_i)$

is the sectional curvature of the plane section spanned by the orthonormal unit vectors e_i and e_j. By our assumption it follows that

$$Q(e_j, e_j) > 0 \quad , \quad j = 1, 2, \ldots, n.$$

Thus from equation (2.2.2) it follows that

$$g(T(e_j, e_j), H) > 0 \; , \quad j = 1, 2, \ldots, n$$

for any local frame $e_1, e_2, \ldots, e_n$. In particular we have $H \neq 0$ throughout the manifold and thus H is globally defined normal vector field.

<u>PROOF OF THE MAIN THEOREM</u>:

Let $F : M \longrightarrow R^{n+2}$ be the isometric immersion of M^n into R^{n+2}.

From the above lemma, we have

$$N = {}^{H}/\| H \|$$

as globally defined nonvanishing unit vectorfield normal to M. Also $g(S_N e_j, e_j) > 0$, $j = 1,2,\ldots,n$. Where S_N is the Weingarten map. Choosing a local frame which diagonalizes S_N, we see that all the principal curvature functions are positive. Now consider the height function

$$h_N : M \longrightarrow R$$

defined by $h_N(x) = \langle F(x), N \rangle$, where $\langle , \rangle$ is the Euclidean innerproduct of R^{n+2}. The hessian H_x of h_N at $x \in M$ is S_N and the index of the non-degenerate critical point of h_N is equal to the number of negative eigen values of h_N.

From the preceeding paragraph, it follows that h_N has no non-degenerate critical point of indices $1,2,\ldots,n-1$. From Morse inequality it follows that

$$H^1(M,R) = \ldots\ldots = H^{n-1}(M,R) = 0$$

This shows that M is simply connected and it being compact, is a homology sphere.

CHAPTER III

TIGHT ISOMETRIC IMMERSION

Immersions of closed manifolds in Euclidean spaces, of minimal total absolute curvature, are called tight. This definition was originated from Chern and Lashoff [9] where they studied the relation between convex hypersurface and absolute curvature. They obtained that if $F : S^n \longrightarrow R^m$ has the minimum total absolute curvature namely 2, then F embeds S^n as a convex hypersurface in euclidean space $R^{n+1} \subseteq R^m$. An important step in the development of the theory was the reformulation of the problem in terms of critical point theory by N.H. Kuiper [22]. He showed that for a given compact manifold M, the infemum of the total curvature $\tau(M,F)$ over all immersions F of M into euclidean spaces is the morse number $\mu(M)$ of M. Moreover this lower bound is attained if, and only if, every non-degenerate linear height function in R^m has $\mu(M)$ critical points on M. Such immersion F is therefore said to have minimum total absolute curvature.

C.S. Chen [7] and [8] studied the tight isometric immersion of a non-negatively curved compact manifold M^n into R^{n+2}, and obtained that the morse number $\mu(M)$ of M cannot exceed 4. Assuming $\mu(M) = 4$ with further condition that no vector of $\bar{M}$ is asymptotic, he proved that M is diffeomorphic to the Riemannian product of spheres. It is therefore interesting to extend this study to non-negatively curved compact manifolds immersed in Euclidean spaces with arbitrary codimension. In this chapter we have obtained similar results for non-negatively curved compact manifolds in Euclidean spaces with arbitrary codimension with additional condition that the normal bundle is flat.

1. PRELIMINARIES: Let M be an n-dimensional Riemannian manifold isometrically immersed in (n+N)-dimensional Riemannian manifold $\bar{M}$ with immersion

$$F : M^n \longrightarrow \bar{M}^{n+N}.$$

Since F is an immersion, F is an embedding on a suitably small neighbourhood of any point $x \in M$. For local calculations, we thus may identify $T_x(M)$ and $F_*(T_x(M))$. In other words locally we identify $T(M)$ and $F_*(T(M))$. Those tangent vectors of $T(\bar{M})$ which are normal to M form the normal bundle $\nu(M)$.

CHAPTER III

TIGHT ISOMETRIC IMMERSION

Immersions of closed manifolds in Euclidean spaces, of minimal total absolute curvature, are called tight. This definition was originated from Chern and Lashoff [9] where they studied the relation between convex hypersurface and absolute curvature. They obtained that if $F : S^n \longrightarrow R^m$ has the minimum total absolute curvature namely 2, then F embeds S^n as a convex hypersurface in euclidean space $R^{n+1} \subseteq R^m$. An important step in the development of the theory was the reformulation of the problem in terms of critical point theory by N.H. Kuiper [22]. He showed that for a given compact manifold M, the infemum of the total curvature $\tau(M,F)$ over all immersions F of M into euclidean spaces is the morse number $\mu(M)$ of M. Moreover this lower bound is attained if, and only if, every non-degenerate linear height function in R^m has $\mu(M)$ critical points on M. Such immersion F is therefore said to have minimum total absolute curvature.

C.S. Chen [7] and [8] studied the tight isometric immersion of a non-negatively curved compact manifold M^n into R^{n+2}, and

Let X, Y be two tangent vector fields on M. Then the shape operator T is given by

$$(3.1.1) \qquad T(X,Y) = \bar{\nabla}_X Y - \nabla_X Y$$

where ∇ and $\bar{\nabla}$ are the covariant derivatives on M and $\bar{M}$ respectively. It is well known that $T(X,Y)$ is a normal vector field on M, and it is symmetric with respect to X and Y. Let f be a normal vector field on M. We write

$$(3.1.2) \qquad \bar{\nabla}_X f = -S_f X + \nabla^{\perp}_X f$$

where $-S_f X$ and $\nabla^{\perp}_X f$ denote the tangential and normal components of $\bar{\nabla}_X f$. Here S_f is linear transformation on $T(M)$ and $\nabla^{\perp}$ is the connexion (covariant derivative) on the normal bundle $\nu(M)$. Then we have,

$$(3.1.3) \qquad \langle S_f X, Y \rangle = \langle T(X,Y), f \rangle$$

Where $\langle \, , \, \rangle$ denotes Riemannian metric on M as well as on $\bar{M}$.

For any 2-plane p spanned by tangent vectors X, Y, we define the sectional curvature of M on p by

$$(3.1.4) \qquad K(p) = \frac{\langle R(X,Y)Y, X \rangle}{\| X \wedge Y \|}$$

Where $R(X,Y)$ is the curvature tensor of M and

(3.1.5) $$\| X \wedge Y \| = \langle X,X \rangle \langle Y,Y \rangle - \langle X,Y \rangle^2$$

NOTE: If M is isometrically immersed in $\bar{M}$, we use bars to distinguish geometric objects on $\bar{M}$. Hence if ∇, R, K are defined on M, then $\bar{\nabla}$, $\bar{R}$, $\bar{K}$ are defined on $\bar{M}$.

The Gauss equation is given by

(3.1.6) $$K(p) = \bar{K}(p) + \frac{\langle T(X,X),T(Y,Y)\rangle - \langle T(X,Y),T(X,Y)\rangle}{\| X \wedge Y \|}$$

for $X,Y \varepsilon T(M)$ and spanning p.

We know that a function ϕ defined over M is said to be a morse function if all of its critical points are non-degenerate. The following theorem is due to M. Morse [28].

PROPOSITION (3.1.1): Given any morse function ϕ on a compact manifold M, there exists a morse function ψ such that

(3.1.7) $$\mu(M, \psi) \leq 2 + \bar{\mu}(M,\phi)$$

Taking infemum on the right hand quantity over all morse function, we get

(3.1.8) $$\mu(M) \leq 2 + \bar{\mu}(M)$$

The morse inequalities which hold for non-degenerate function ϕ on a compact manifold M, gives

(3.1.9) $$\mu_k(M,\phi) \geq \mu_k(M) \geq b_k(M,F)$$

Where $b_k(M,F)$ is the dimension of k-th homology group of M with coefficient in the field F. Let $F : M^n \longrightarrow R^{n+N}$ be an isometric immersion of a compact connected n-manifold into the Euclidean space R^{n+N}.

Let A be the set of unit normals f such that S_f is definite and B be the set of unit normals f such that S_f is not definite. By Morse inequalities,

(3.1.10) $$\frac{1}{|S(1)^{n+N-1}|} \int_{f\varepsilon\nu(M)} |\det S_f| \; |\omega| \geq \mu(M).$$

(3.1.11) $$\frac{1}{|S(1)^{n+N-1}|} \int_{f\varepsilon B} |\det S_f| \; |\omega| \geq \bar{\mu}(M).$$

Where $|\omega|$ is the volume element of $\nu(M)$ and $|S(1)^{n+N-1}|$ is the volume of the unit (n+N-1)-sphere $S(1)^{n+N-1}$. We also know that

(3.1.12) $$\frac{1}{|S(1)^{n+N-1}|} \int_{f\varepsilon\nu(M)} |\det S_f| \; |\omega| = \tau(M),$$

is the total absolute curvature of the isometric immersion F.
If we put $d = \tau(M) - \mu(M)$, then F will be tight if $d = 0$.

2. TIGHT ISOMETRIC IMMERSION OF M^n INTO R^{n+N}

Let M be an n-dimensional compact, non-negatively curved manifold isometrically immersed in R^{n+N} and assume that all the sectional curvatures of M are non-negative. Let $F : M \longrightarrow R^{n+N}$ be the required immersion. Since all the sectional curvatures of M are non-negative, by Gauss equation, we get

$$(3.2.1) \qquad \langle T(X,X),T(Y,Y) \rangle - \langle T(X,Y),T(X,Y) \rangle \geq 0$$

Where X,Y are the tangent vectors of M.

$$\Longrightarrow \qquad \langle T(X,X), T(Y,Y) \rangle \geq 0$$

This means that for all tangent vectors X,Y, $T(X,X)$ & $T(Y,Y)$ make an acute angle with each other. This suggests the decomposition of the normal space R^N of M at every point $x \in M$ into 2^N regions (similar to 2^2 quadrants of R^2 or 2^3 octants in R^3). Therefore at every point of M, we can choose a coordinate system in the normal space such that $T(X,X)$ lies in the first of the 2^N regions. Obviously every vector of this region is the linear combination of the vectors $f_1, f_2, \ldots, f_N$ with positive coefficients where each f_i is a unit vector along the positive coordinate axes of R^N, the normal space at a point of M. We shall call this as region I and say briefly that it is positively spanned by $f_1, f_2, \ldots, f_N$. Now $f \in \mathcal{V}(M)$ at any given point can be written as

$$f = \cos \theta_1 f_1 + \cos \theta_2 f_2 + \ldots + \cos \theta_N f_N. \tag{3.2.2}$$

Where θ_i is the angle which f makes with the positive i-th axis. Now we have the following analysis.

<u>Case1</u>: Whenever $f \in I$, $T(X,X)$ & f make an acute angle with each other

i.e. $\langle T(X,X), f \rangle \geq 0$

or $\langle S_f X, X \rangle \geq 0$

This means <u>S_f is positive definite whenever $f \in I$</u>

<u>Case 2</u>: Let us call the region positively spanned by $(-f_i)$ as region I^*, if $f \in I^*$, then

$$f = \cos \theta_1(-f_1) + \ldots + \cos \theta_N (-f_N)$$

$$= - \cos \theta_1 f_1 - \ldots - \cos \theta_N f_N.$$

and $T(X,X) = \cos \Theta_1' f_1 + \ldots + \cos \Theta_N' f_N$.

Where $\cos \Theta_i' \geq 0$, as $0 \leq \Theta_i' \leq \pi/2$, $i = 1,2,\ldots,N$

$\Longrightarrow \langle T(X,X), f \rangle = -\cos \Theta_1 \cos \Theta_1' - \ldots - \cos \Theta_N \cos \Theta_N'$.

i.e. $\langle T(X,X), f \rangle \leq 0$

or $\langle S_f(X), X \rangle \leq 0$

This means, <u>S_f is negative definite whenever $f \in I^*$</u>

<u>Case 3</u>: Consider the region positively spanned by

$f_1, f_2, \ldots, f_i, -f_{i+1}, -f_{i+2}, \ldots, -f_N$, then

$f = \cos \Theta_1 f_1 + \ldots + \cos \Theta_i f_i - \cos \Theta_{i+1} f_{i+1} - \ldots - \cos \Theta_N f_N$.

Now,

$$\langle S_f(X), X\rangle = \langle T(X,X), f\rangle = \cos \Theta_1 \cos \Theta_1' + \ldots + \cos \Theta_i \cos \Theta_i'$$

$$- \cos \Theta_{i+1} \cos \Theta_{i+1}' - \ldots - \cos \Theta_N \cos \Theta_N'.$$

Which could be positive, negative or zero depending on i.

Hence, <u>S_f is indefinite in the region other than I and I*</u>.

Let us write, for $f \in \nu(M)$

(3.2.3) $f = f(\Theta_1, \Theta_2, \ldots, \Theta_N) = \cos \Theta_1 f_1 + \ldots + \cos \Theta_N f_N$.

and

(3.2.4) $S_{f(\Theta_1,\Theta_2,\ldots,\Theta_N)} = S_{(\Theta_1,\Theta_2,\ldots,\Theta_N)}$

$$= \cos \Theta_1 S_{f_1} + \ldots + \cos \Theta_N S_{f_N}.$$

It follows from Case 1, 2 and 3 that $S_{(\Theta_1,\Theta_2,\ldots,\Theta_N)}$ is an indefinite linear transformation on the tangent space $T_x(M)$ of M at every point x. Whereas S_{f_i}'s are positive semi definite. It is easy to observe that atleast one of the S_{f_i} is a positive definite, symmetric $(n \times n)$-matrix. By above analysis A is contained in region I and I*, whereas B is contained in the rest of the 2^N-2 regions. $f(\Theta_1,\ldots,\Theta_N)$ is obviously in the region I whenever $o \leq \Theta_i \leq \pi/2$ for all i and then $S_{(\Theta_1,\ldots,\Theta_N)}$ is positive definite. Our aim is to compare the value of $|\det S_{(\Theta_1,\ldots,\Theta_N)}|$ when $S_{(\Theta_1,\ldots,\Theta_N)}$ is positive definite (or negative definite) with that when it is indefinite. Infact we prove the following lemma.

<u>LEMMA</u> (3.2.1): If M is a compact Riemannian manifold of non-negative curvature which admits an isometric immersion in R^{n+N} with flat normal connexion, then for all normal vector field f and f'

$$S_f \, S_{f'} = S_{f'} \, S_f \tag{3.2.5}$$

and

$$|\det S_{P(\Theta_1,\ldots,\Theta_N)}| \leq |\det S_{(\Theta_1,\ldots,\Theta_N)}| \tag{3.2.6}$$

Where S_f denotes the Weingarten map and $P(\Theta_1,\ldots,\Theta_N)$ is obtained by replacing atleast one of the Θ_i (and not all Θ_i's) by $(\pi-\Theta_i)$ in $\Theta_1,\ldots,\Theta_N$.

<u>PROOF</u>: Let $F : M^n \longrightarrow R^{n+N}$ be the isometric immersion of M into R^{n+N}. If $f_1,\ldots,f_N$ is the orthonormal basis of the normal space at some point, then for X,Y tangent vectors of M, we have the Ricci equation.

(3.2.7) $\bar{R}(X,Y,f_i,f_j) = R\,(X,Y,f_i,f_j) - g([S_{f_i},S_{f_j}]X,Y)$

which gives

$$g([S_{f_i},S_{f_j}]X,Y) = 0\,, \quad \text{as } \bar{R} \text{ and } R = 0$$

This means,

$$S_{f_i}\,S_{f_j} = S_{f_j}\,S_{f_i}$$

which proves (3.2.5).

Since S_{f_i}'s are symmetric (n x n)-real matrices which commute with each other and therefore all S_{f_i}'s can be diagonalized simulateneously. Let Q be the matrix which diagonolizes all S_{f_i}'s simultaneously, then

$$Q^{-1}S_{(\Theta_1,\ldots,\Theta_N)}\,Q = \cos\Theta_1\,Q^{-1}\,S_{f_1}Q + \ldots + \cos\Theta_N\,Q^{-1}S_{f_N}Q.$$

or $Q^{-1}S_{(\Theta_1,\ldots,\Theta_N)}\,Q = \text{Cos}\,\Theta_1\begin{pmatrix}\lambda_1^1 & & & \\ & \lambda_2^1 & & \\ & & \ddots & \\ & & & \lambda_n^1\end{pmatrix} + \ldots + \text{Cos}\,\Theta_N\begin{pmatrix}\lambda_1^N & & & \\ & \lambda_2^N & & \\ & & \ddots & \\ & & & \lambda_n^N\end{pmatrix}.$

Where some of the eigenvalues λ_j^i may be zero, but there exists atleast one matrix which has all positive eigen values. Taking determinant of both the sides, we get

$$\det S_{(\Theta_1,\ldots,\Theta_N)} = (\text{Cos}\,\Theta_1\,\lambda_1^1 + \text{Cos}\,\Theta_2\,\lambda_1^2 + \ldots + \text{Cos}\,\Theta_N\,\lambda_1^N) \ldots$$
$$\ldots (\text{Cos}\,\Theta_1\lambda_n^1 + \ldots + \text{Cos}\,\Theta_N\,\lambda_n^N).$$

Where $\lambda_j^i \geq 0$, but every bracket contains some non zero terms. Noting a simple fact that $\text{Cos}\,(\pi - \Theta) = -\,\text{Cos}\,\Theta$ and the inequality

$$|\,\Sigma\varepsilon_j a_j| \leq |\,\Sigma a_j|$$

for non negative numbers a_j, where $\varepsilon_j = 1$ or -1, we easily get,

$$|\det S_{P(\Theta_1,\ldots,\Theta_N)}| \leq |\det S_{(\Theta_1,\ldots,\Theta_N)}|.$$

Q.E.D.

<u>LEMMA</u> (3.2.2): Let M^n be a non-negatively curved compact manifold such that all of its sectional curvatures are also non negative. Then the isometric immersion of M into R^{n+N} with flat normal connexion satisfies

(3.2.8) $$\int_{f\in B} |\det S_f|\ |\omega| \leq (2^{N-1}-1) \int_{f\in A} |\det S_f|\ |\omega|$$

PROOF: Since A is contained in the region I and I* of the normal bundle and B is contained in the rest of the 2^N-2 regions. Using lemma (3.2.1), we get,

$$2\int_{f\in B} |\det S_f|\ |\omega| \leq (2^N-2) \int_{f\in A} |\det S_f|\ |\omega|$$

Hence the lemma is proved.

THEOREM (3.2.1): If M^n is a compact Riemannian manifold of non-negative curvature which admits an isometric immersion in R^{n+N}, with flat normal connexion, then

(3.2.9) $$\mu(M) \leq (2^{N-1}-1)\ (d+2) + 2$$

PROOF: From (3.1.10) and (3.1.11), we have

$$\frac{1}{|S(1)^{n+N-1}|} \int_{f\in B} |\det S_f|\ |\omega| \geq \bar{\mu}(M) \geq \mu(M) - 2$$
$$= \tau(M) - d - 2$$

$$\Longrightarrow \frac{1}{|S(1)^{n+N-1}|} \int_{f\in B} |\det S_f|\ |\omega| \geq \frac{1}{|S(1)^{n+N-1}|} \int_{f\in\nu(M)} |\det S_f|\ |\omega| - d - 2$$

or $$\frac{1}{|S(1)^{n+N-1}|} \int_{f\in B} |\det S_f|\ |\omega| \geq \frac{1}{|S(1)^{n+N-1}|} \int_{f\in A} |\det S_f|\ |\omega|$$
$$+ \frac{1}{|S(1)^{n+N-1}|} \int_{f\in B} |\det S_f|\ |\omega| - d - 2$$

or $$d+2 \geq \frac{1}{|S(1)^{n+N-1}|} \int_{f\varepsilon A} |\det S_f| \, |\omega|.$$

Using (3.2.8), we get

$$d+2 \geq \frac{1}{(2^{N-1}-1)} \times \frac{1}{|S^{n+N-1}(1)|} \int_{f\varepsilon B} |\det S_f| \, |\omega|$$

or $$d+2 \geq \left(\frac{1}{2^{N-1}-1} \right) \bar{\mu}(M)$$

$$\geq \frac{1}{2^{N-1}-1} (\mu(M) - 2).$$

or

(3.2.10) $$\mu(M) \leq (2^{N-1}-1)(d+2) + 2$$

Immersion is tight if and only if $d = o$, Thus we have.

<u>LEMMA</u> (3.2.3): If M^n is a compact Riemannian manifold of non-negative curvature which admits a tight isometric immersion in R^{n+N}, then

(3.2.11) $$\mu(M) \leq 2^N.$$

<u>LEMMA</u> (3.2.4): Given an isometric immersion of a non-negative curved manifold M into R^{n+N}. If all sectional curvatures are strictly positive at some point, then

$$(3.2.12) \qquad \int_{f\varepsilon B} |\det S_f|\ |\omega| < (2^{N-1}-1) \int_{f\varepsilon A} |\det S_f|\ |\omega|$$

PROOF: If all the sectional curvatures are strictly positive at a point $x \varepsilon M$, then it would be strictly positive in a compact neighbourhood S of x. Again by Gauss theorem

$$< T(X,X),\ T(Y,Y) > > 0$$

for any two nonzero $X \neq Y$ tangent to M at a point of S.

As before we choose the coordinate system in the normal space, such that $T(X,X)$ lies in the interior of the region I. Hence A is contained in the interior of the region I and I* and B is not only contained in the rest of the other regions but also occupies some portion of the region I and I*.

This means,

$$|\det S_{P(\theta_1,\ldots\theta_N)}| < |\det S_{(\theta_1,\ldots,\theta_N)}|$$

for all $o < \theta_i < \pi/2$ is satisfied in S. Now extending the above strict inequality on the whole manifold, we get

$$\int_{f\varepsilon B} |\det S_f|\ |\omega| < (2^{N-1}-1) \int_{f\varepsilon A} |\det S_f|\ |\omega|.$$

Q.E.D.

<u>LEMMA</u> (3.2.5): If M^n is a compact Riemannian manifold of non-negative curvature with strictly positive sectional curvature at some point, then

(3.2.13) $$\mu(M) < (2^{N-1}-1)\ (d+2) + 2$$

Where $d = \tau(M) - \mu(M)$.

<u>PROOF</u>: The proof directly follows from Theorem (2.2.1) just by changing the some of the inequalities into sharp inequalities and we have

$$\mu(M) < (2^{N-1}-1)\ (d+2) + 2 .$$

Hence we have the following result:

<u>THEOREM</u> (3.2.2): Let M be a compact Riemannian manifold of non-negative curvature with strictly positive sectional curvature at some point. If M^n admits a tight isometric immersion into R^{n+N}, then

(3.2.14) $$\mu(M) \leq 2^N-1.$$

Q.E.D.

CHAPTER IV

SPECIAL CASE OF TIGHT IMMERSION

In the previous chapter, we obtained an estimate of the critical points of the morse function on a compact non-negatively curved manifold M^n, admitting a tight isometric immersion in the Euclidean space R^{n+N}. We found that $\mu(M) \leq 2^N$. In the case when M has strictly positive sectional curvature at some point, then $\mu(M) \leq 2^N-1$. This explains that if $\mu(M) = 2^N$, then $T(X,X)$ should occupy the full region I because otherwise $\mu(M) \leq 2^N-1$. Therefore in this situation we are in a position to find an splitting of the tangent space such that each factor is integerable and totally geodesic. This helps in establishing the fact that M is the product of spheres.

1. PRELIMINARIES:

Let M be a C^∞-manifold of dimension n. We need the following notions in this chapter.

DEFINITION (4.1.1): A distribution S of dimension r on M is an assignment to each point x of M an r-dimensional subspace S_x of $T_x(M)$. It is called differentiable if every point x has a neighbourhood U and r differentiable vector-fields on U, say $X_1, \ldots, X_r$ which form a basis of S_y at every $y \varepsilon U$. The set $X_1, \ldots, X_r$ is called a <u>local basis</u> for the distribution S in U. A vector field X is said to belong to S, if $X_x \varepsilon S_x$ for all $x \varepsilon M$.

<u>DEFINITION</u> (4.1.2): S is called involutive if $[X,Y] \varepsilon S$ whenever $X,Y \varepsilon S$.

<u>DEFINITION</u> (4.1.3): A connected submanifold N of M is called an <u>integeral manifold</u> of the distribution S if $F_*(T_x(M)) = S_x$ for all $x \varepsilon N$ where F is the embedding of N into M.

<u>DEFINITION</u> (4.1.4): If there is no other integeral manifold of S which contains N, then N is called <u>maximal integeral manifold</u> of S.

Now we state the famous Frobeneous theorem.

<u>THEOREM</u> (4.1.1): Let S be an involutive distribution on a manifold M. Through every point $x \varepsilon M$, there passes a unique maximal integeral manifold $N(x)$ of S. Any integeral manifold through x is an open submanifold of $N(x)$.

DEFINITION (4.1.5): If M is a C^∞-manifold, a (usually) disconnected r-dimensional submanifold N of M is called a foliation if every point of M is in (some component of) N, and if around every point $x \in M$, there is a (cubic) coordinate system (U,ϕ) with

$$\phi(U) = (-\varepsilon, \varepsilon)X,\ldots,X(-\varepsilon,\varepsilon)$$

Such that the components of $N \cap U$ are the sets of the form

$$\{y \in U : \phi^{k+1}(y) = a^{k+1},\ldots,\phi^n(y) = a^n\}, \quad |a^i| < \varepsilon$$

Each component of N is called a leaf of the foliation. Notice that two distinct components of $N \cap U$ might belong to the same leaf of foliation.

Now at the end of this article, we state the following results [25] and [11], which we use in this chapter.

MAYER'S THEOREM (4.1.2): If M is complete and $K(X,X) \geq \frac{n-1}{r^2} > 0$ for all unit vectors X, then M is compact with diameter $< \pi r$.

THEOREM (4.1.3): Assume all second quadretic forms of the immersion $x : M^n \longrightarrow R^{n+N}$ to be semi definite and M is compact, oriented n-dimensional Riemannian manifold. Then $x(M)$ belongs to a linear subvariety R^{n+1} of R^{n+N} and $x : M \longrightarrow R^{n+1}$ embeds M as the boundary of convex body, in particular, M is homeomorphic to a sphere.

2. <u>SPECIAL CASE OF TIGHT IMMERSION</u>:

As in the previous chapter, let M^n be a compact Riemannian manifold of non-negative curvature isometrically immersed in R^{n+N}. Further we assume the following:

i) $\tau(M) = \mu(M) = 2^N$.

ii) There are no asymptotic vectors.

In this special case, the $T(X,X)$ which is never zero unless $X = 0$ must range over the full region I, otherwise $\mu(M) \leq 2^N-1$. Our aim is to get an splitting of the tangent space so that each factor is integerable.

<u>DEFINITION</u> (4.2.1): Let $f_1,\ldots,f_N$ be the orthonormal basis of the normal space, then we define

$$U_i = \{ X \in T(M)/T(X,Y) // f_i \quad \text{for all} \quad Y \in T(M) \}$$

<u>LEMMA</u> (4.2.1): U_i's are the subspaces of the tangent space $T(M)$ at each point $x \in M$, and that

$$T(M) = \bigoplus_{i=1}^{n} U_i .$$

PROOF: Let $X, Y \varepsilon U_i$, therefore

$T(\alpha X + \beta Y, Z)$ is parallel to f_i, this means $\alpha X + \beta Y \varepsilon U_i$. This proves that U_i is the subspace of the tangent space for each i $(i = 1, \ldots, N)$. Furthermore we observe that for each scalar $\alpha_i \varepsilon R$, there exists $X_i \varepsilon U_i$ such that $T(X_i, Y) = \alpha_i f_i$. Let X be tangent vector of M and for fixed Y,

$$\begin{aligned} T(X,Y) &= \alpha_1 f_1 + \ldots + \alpha_N f_N \\ &= T(X_1,Y) + T(X_2,Y) + \ldots + T(X_N,Y) \\ &= T(X_1 + \ldots + X_N, Y) \end{aligned}$$

$$\Longrightarrow \quad X = X_1 + X_2 + \ldots + X_N.$$

This means for each tangent vector X of M, there exists tangent vectors $X_1, X_2, \ldots, X_N$ such that $X_i \in U_i$ and

$$X = X_1 + X_2 + \ldots + X_N$$

$$\Longrightarrow \quad T(M) \subseteq U_1 + U_2 + \ldots + U_N$$

Take $X_i \varepsilon U_i$ and suppose $X_i \varepsilon U_j$ also

$\Longrightarrow$ $T(X_i, Y) \mathbin{//} f_i$ for all Y and $T(X_i, Y) \mathbin{//} f_j$

But $f_i \perp f_j$ $\Longrightarrow$ $T(X_i, Y) \perp T(X_i, Y)$

$$\Longrightarrow \quad T(X_i, Y) = 0, \qquad \text{for all} \quad Y$$

$$\Longrightarrow \quad X_i = 0 \quad \Longrightarrow \quad U_i \cap U_j = \{0\}$$

$$\Longrightarrow \quad T(M) = \bigoplus_{i=1}^{N} U_i$$

Q.E.D.

Hence under the assumption (i) and (ii), the tangent space $T(M)$ at each point splits uniquely as

$$T(M) = U_1 \oplus U_2 \oplus \cdots \oplus U_N.$$

The distributions U_i's are differentiable follows from the fact that the eigen spaces corresponding to distinct non-zero eigen values of the quadratic form are differentiable. The details in the case of codimension two have been worked out by C.S. Chen [7].

<u>LEMMA</u> (4.2.2): U_i's are integerable.

<u>PROOF</u>: Our aim, here, is to show that for $X, Y \in U_i$, $[X,Y] \in U_i$ Take $X, Y \in U_1$, then

$$\langle T(X,Y), f_i \rangle = 0 \qquad \text{for all} \quad Y \quad \text{and} \quad i = 2,3,\ldots,N.$$

$$\Longrightarrow \quad \langle \bar{\nabla}_X f_i, Y \rangle = 0$$

$$\Longrightarrow \quad \bar{\nabla}_X f_i \in \nu\,(M).$$

This means,

$$\bar{\nabla}_X f_i = \alpha_1 f_1 + \ldots + \widehat{\alpha_i f_i} + \ldots + \alpha_N f_N.$$

Where a hat over a term means it is left out. Using the flatness of the Euclidean space, we compute for $X, Y \in U_1$

$$0 = \bar{\nabla}_X \bar{\nabla}_Y f_i - \bar{\nabla}_Y \bar{\nabla}_X f_i - \bar{\nabla}_{[X,Y]} f_i, \quad \text{for all } i = 2, 3, \ldots, N.$$

$$\Longrightarrow \quad \bar{\nabla}_{[X,Y]} f_i = \bar{\nabla}_X (\beta_1 f_1 + \ldots + \widehat{\beta_i f_i} + \ldots + \beta_N f_N)$$

$$- \bar{\nabla}_Y (\alpha_1 f_1 + \ldots + \widehat{\alpha_i f_i} + \ldots + \alpha_N f_N), \quad \text{for all } i = 2, \ldots, N$$

$$= [X(\beta_1) - Y(\alpha_1)] f_1 + \ldots + \widehat{[X(\beta_i) - Y(\alpha_i)] f_i} + \ldots + [X(\beta_N) - Y(\alpha_N)] f_N$$

$$+ \beta_1 \bar{\nabla}_X f_1 + \ldots + \widehat{\beta_i \bar{\nabla}_X f_i} + \ldots + \beta_N \bar{\nabla}_X f_N - \alpha_1 \bar{\nabla}_Y f_1 - \ldots - \widehat{\alpha_i \bar{\nabla}_Y f_i}$$

$$- \ldots - \alpha_N \bar{\nabla}_Y f_N.$$

Taking inner product with $Z \in T(M)$ in both the sides, we get

$$\langle \bar{\nabla}_{[X,Y]} f_i, Z \rangle = \beta_1 \langle \bar{\nabla}_X f_1, Z \rangle + \ldots + \widehat{\beta_i \langle \bar{\nabla}_X f_i, Z \rangle} + \ldots + \beta_N \langle \bar{\nabla}_X f_N, Z \rangle$$

$$- \alpha_1 \langle \bar{\nabla}_Y f_1, Z \rangle - \ldots - \widehat{\alpha_i \langle \bar{\nabla}_Y f_i, Z \rangle} - \ldots - \alpha_N \langle \bar{\nabla}_Y f_N, Z \rangle$$

$$(4.2.1) \quad \langle \bar{\nabla}_{[X,Y]} f_i, Z \rangle = \alpha_1 \langle T(Y,Z), f_1 \rangle + \ldots + \widehat{\alpha_i \langle T(Y,Z), f_i \rangle} + \ldots$$

$$+ \alpha_N \langle T(Y,Z), f_N \rangle - \beta_1 \langle T(X,Z), f_1 \rangle - \ldots$$

$$- \widehat{\beta_i \langle T(X,Z), f_i \rangle} - \ldots - \beta_N \langle T(X,Z), f_N \rangle.$$

Case 1: Let $Z \in U_i$.

In this case $T(Z,W) \parallel f_i$ for all W, therefore the inner product of $T(Z,W)$ with all normals f_j is zero except the i-th normal f_i. But the i-th term in the right hand side of the above equation (4.2.1) is missing, therefore for $Z \in U_i$

$$\langle \bar{\nabla}_{[X,Y]} f_i, Z \rangle = 0.$$

Case 2: Let $Z \in U_1$, then

$$\langle \bar{\nabla}_{[X,Y]} f_i, Z \rangle = \langle T(Z,[X,Y]), f_i \rangle = 0$$

as $X,Y,Z \in U_1$ and $2 \leq i \leq N$.

Case 3: Let $Z \in U_j$ $(j \neq 1,i)$, then, equation (3.2.1) implies

$$\langle \bar{\nabla}_{[X,Y]} f_i, Z \rangle = \langle T(Z, \alpha_j Y - \beta_j X), f_j \rangle$$

$$= \langle T(\alpha_j Y - \beta_j X, Z), f_j \rangle$$

$$= \langle T(Y, \alpha_j Z), f_j \rangle - \langle T(X, \beta_j Z), f_j \rangle$$

Since $X,Y \in U_1$, both the term in the right hand side of the above equation are zero. Hence

$$\langle \bar{\nabla}_{[X,Y]} f_i, Z \rangle = 0 \quad \text{for } X,Y \in U_1 \text{ and } Z \in T(M) \text{ and}$$

$$\text{for all } i = 2,3,\ldots,N.$$

$\Longrightarrow \quad \langle f_i, \bar{\nabla}_{[X,Y]}Z \rangle = 0,$

or $\quad \langle f_i, \nabla_{[X,Y]}Z + T([X,Y],Z) \rangle = 0,$

or $\quad \langle f_i, T([x,Y],Z) \rangle = 0$

or $\quad T([X,Y],Z) \perp f_i,$ for all $i = 2,3,\ldots,N.$

$\Longrightarrow \quad T([X,Y],Z) \,//\, f_1.$

$\Longrightarrow \quad [X,Y] \in U_1.$

Hence U_1 is integerable. Similarly we can show that $U_2, U_3, \ldots, U_N$ are also integerable.

Q.E.D.

Now we can use Frobenius theorem to obtain a foliation of M by leaves $U_1, U_2, \ldots, U_N$ of codimensions $p_1, p_2, \ldots, p_N$ respectively.

<u>LEMMA</u> (3.2.3): The leaves are totally Geodesic submanifold of M.

<u>PROOF</u>: To prove the lemma, we show that the shape operator of the leaves are zero. Let $X,Y \in U_1$, then we evaluate

$$\langle \bar{\nabla}_{\nabla_X Y} f_i, Z \rangle \text{ for } Z \in T(M) \text{ and } i = 2,3,\ldots,N.$$

$$\langle \bar{\nabla}_{\nabla_X Y} f_i, Z \rangle = \langle S_{f_i} \nabla_X Y, Z \rangle = \langle T(\nabla_X Y, Z), f_i \rangle$$

If $Z \in U_1$, then $T(Z, \nabla_X Y) \,//\, f_1$

$\Longrightarrow \quad \langle T(Z, \nabla_X Y), f_i \rangle = 0$

Hence for $Z \in U_1$, $\langle \bar{\nabla}_{\nabla_X Y} f_i, Z \rangle = 0$,

If $Z \in U_j$ $(j \neq 1)$, then

$$\langle \bar{\nabla}_{\nabla_X Y} f_i, Z \rangle = - \langle \bar{\nabla}_{\nabla_X Y} Z, f_i \rangle$$

$$= - \langle \bar{\nabla}_Z \nabla_X Y, f_i \rangle$$

$$= - \langle \bar{\nabla}_Z (\bar{\nabla}_X Y - T(X,Y)), f_i \rangle$$

$$= - \langle \bar{\nabla}_Z (\bar{\nabla}_X Y), f_i \rangle + \langle \bar{\nabla}_Z (T(X,Y)), f_i \rangle$$

i.e.

$$(4.2.2) \quad \langle \bar{\nabla}_{\nabla_X Y} f_i, Z \rangle = - \langle \bar{\nabla}_X (\bar{\nabla}_Z Y) - \bar{\nabla}_{[X,Z]} Y, f_i \rangle + \langle \bar{\nabla}_Z (T(X,Y), f_i \rangle$$

Now $\langle \bar{\nabla}_{[X,Z]} Y, f_i \rangle = \langle Y, \bar{\nabla}_{[X,Z]} f_i \rangle = \langle Y, S_{f_i} [X,Z] \rangle$

$$= \langle Y, S_{f_i} [X,Z] \rangle = \langle T([X,Z],Y), f_i \rangle$$

$$(4.2.3) \quad \langle \bar{\nabla}_{[X,Z]} Y, f_i \rangle = \langle T(Y,[X,Z]), f_i \rangle = 0, \quad \text{as } Y \in U_1$$

Also $\langle \bar{\nabla}_X(\bar{\nabla}_Z Y), f_i \rangle = \langle \bar{\nabla}_X(\nabla_Z Y), f_i \rangle + \langle \bar{\nabla}_X(T(Z,Y)), f_i \rangle$

therefore,

(4.2.4) $\langle \bar{\nabla}_X(\bar{\nabla}_Z Y), f_i \rangle = - \langle \nabla_Z Y, \bar{\nabla}_X f_i \rangle - \langle T(Z,Y), \bar{\nabla}_X f_i \rangle$

Now for $X \varepsilon U_1$, consider

$0 = \langle T(X,Y), f_i \rangle$, $i = 2,3,\ldots,N.$

or $\langle S_{f_i} X, Y \rangle = 0,$

$\Longrightarrow \langle \bar{\nabla}_X f_i, Y \rangle = 0,$ for all Y.

$\Longrightarrow \bar{\nabla}_X f_i = 0$

Using this fact in equation (4.2.4), we get

(4.2.5) $\langle \bar{\nabla}_X(\bar{\nabla}_Z Y), f_i \rangle = 0$

Now $\langle \bar{\nabla}_Z(T(X,Y)), f_i \rangle = \langle \bar{\nabla}_Z \alpha_1 f_1, f_i \rangle = -\alpha_1 \langle f_1, \bar{\nabla}_Z f_i \rangle$.

Since $\bar{\nabla}_Z f_i = 0$ for $Z \varepsilon U_j$ $(j \neq i)$, we get

(4.2.6) $\langle \bar{\nabla}_Z(T(X,Y)), f_i \rangle = 0$

Therefore from equations (4.2.2), (4.2.3), (4.2.5) and (4.2.6), we get

$$\langle \bar{\nabla}_{\bar{\nabla}_X Y} f_i, Z \rangle = 0 \quad \text{for all } Z \varepsilon T(M).$$

$\Longrightarrow \quad \langle S_{f_i} \nabla_X Y, Z \rangle = 0, \quad \text{as} \quad \nabla^{\perp}_{\nabla_X Y} f_i = 0$

or $\quad \langle T(\nabla_X Y, Z), f_i \rangle = 0, \quad$ for all $\quad Z \in T(M)$

or $\quad T(\nabla_X Y), Z) \perp f_i$

$\Longrightarrow \quad T(\nabla_X Y, Z) \; // \; f_1.$

$\Longrightarrow \quad \nabla_X Y \in U_1.$

This proves that U_1 is totally Geodesic similarly we can show that all U_i's are totally geodesic. This proves the lemma.

Next three lemmas follow exactly on the same lines as in the case of codimension two obtained by C.S. Chen [7]. Nevertheless for the sake of completeness we write down the proofs for codimension N.

<u>LEMMA</u> (4.2.4): Each leaf is compact and belongs to a linear subvariety of one higher dimension. Infact the leaf bounds a strictly convex set in a linear subvariety.

<u>PROOF</u>: Let X be a tangent vector of M such that $\|X\| = 1$, then $\|T(X,X)\| \neq 0$. Hence there is a $\delta > 0$ such that $\|T(X,X)\| \geq \delta$ for all X such that $\|X\| = 1$. This fact

together with the previous lemma, implies that all the sectional curvatures of the leaves are $\geq \delta^2$. Now applying Mayer's theorem (4.1.2), all the leaves are compact. Let $\bar{T}$ denotes the shape operator of a leaf L in R^{n+N}. Since $L \subseteq \mathcal{U}_i$ is totally geodesic in M, therefore $\bar{T}(X,Y) = T(X,Y)$ is parallel to f_i $\forall$ X,Y tangential to L. Therefore all second fundamental forms $\bar{S}$ of L, considered to be immersed in R^{n+N}, are semi definite. Hence from Theorem (4.1.3) it follows that, L is embedded as the boundary of a convex set in some linear sub-variety of one higher dimension.

<u>LEMMA</u> (4.2.5): Each leaf in $\mathcal{U}_i$ intersect each leave in $\mathcal{U}_j$ ($j \neq i$) only on ce whereas no leaf in the same foliation intersect.

<u>PROOF</u>: We use the quotient topology on the collection of leaves. Let L_i be a leaf in $\mathcal{U}_i$. To prove that each leaf in $\mathcal{U}_j$ intersect L_i, let $\mathcal{V}_j \subseteq \mathcal{U}_j$ be the set of leaves which intersect L_i. Since L_i is compact, $\mathcal{V}_j$ is closed in $\mathcal{U}_j$. By transverality of L_i with every leaf in $\mathcal{U}_j$, $\mathcal{V}_j$ is open. Since $\mathcal{U}_j$ is connected $\mathcal{V}_j = \mathcal{U}_j$. Also at each point F(x), the

hyperplane $f_i^{\perp}$ spanned by f_j's $(j \neq 1)$ and $U_1, U_2, \ldots, U_N$ is a supporting hyperplane [7]. Where a supporting hyperplane of $F(M)$ at $F(x)$ is a hyperplane through $F(x)$ such that $F(M)$ lies in a closed half space determined by the hyperplane.

We have just seen that $F(L_i)$ intersect with $F(L_j)$ for $L_i \in \mathcal{U}_i$ and $L_j \in \mathcal{U}_j$. Now we want to show that they intersect only once. For this on the contrary suppose $F(L_i)$ intersect $F(L_j)$ at $F(x)$ and $F(y)$ with $x \neq y$. Then $f_j^{\perp}$ at $F(x)$ supports $F(M)$ and contains $F(L_j)$, hence contains $F(y)$. Thus $f_j^{\perp}$ should contain all $F(L_i)$ in order to be a supporting hyperplane. This shows $f_j \in f_j^{\perp}$ which is a contradiction, hence the leaves should intersect only once.

Finally we show that no leaves in the same foliation intersect. For this suppose L_i and L_i' are distinct leaves in $\mathcal{U}_i$. Such that $F(L_i)$, $F(L_i')$ intersect at $F(x)$ with $x \in L_i$. Now the leaf L_j which passes through x must intersect L_i' at y which contradicts the fact that each leaf in $\mathcal{U}_i$ intersect each leaf in $\mathcal{U}_j$. This proves the lemma completely.

<u>REMARK</u>: From the above, lemma it is also observed that F is

an embedding.

LEMMA (4.2.6): M is diffeomorphic to the product of spheres $S^{p_1} \times S^{p_2} \times, \ldots, \times S^{p_N}$.

PROOF: Each leaf is diffeomorphic to the sphere by lemma (4.2.4). Now consider L_i and L_i' in the same foliation $\mathcal{U}_i$ for each $p \in L_i$, let L_j be the complementary leaf through p and $p' = L_i' . L_j$ be the intersection of L_i' and L_j. Consider the map

$$\phi_{L_i L_i'} : p \longrightarrow p'.$$

These maps are well defined by previous lemma. Since the complementary leaves are transversal to each other, therefore through each point $p \in L_i$, there exists a cubic coordinate system which is centred at m with coordinates $x_1, x_2, \ldots, x_n$ such that the slices

$$x_i = \text{Constant}, \quad \text{for } 1 \leq i \leq p_i$$

represents parts of the leaves of $\mathcal{U}_i$ in the cube.

By covering L_i with finite number of such cubes, we have shown that $\phi_{L_i L_i'}$ is a diffeomorphism if L_i' is sufficiently close to L_i. Our claim follows by using compactness of L_j.

Finally take two fixed complementary leaves L_{i_o} and L_{j_o}. For each $m \varepsilon M$, let L_i' and L_j' be the complementary leaves through m. Then the required diffeomorphism is given by

$$m \longrightarrow (L_{i_o} \cdot L_1', \ldots, L_{i_o} \cdot L_{i_o}', \ldots, L_{i_o} \cdot L_N', L_{j_o} \cdot L_{i_o})$$

which proves the lemma.

Q.E.D.

Summarizing the results obtained in this section, we state the main theorem of this chapter as follows:

<u>THEOREM</u> (4.2.1): Let $F : M^n \longrightarrow R^{n+N}$ be a tight isometric immersion of a compact manifold of non-negative curvature with flat normal connexion. Assume $\tau(M) = \mu(M) = 2^N$, and there are no asymptotic vectors. Then F is an embedding and M is deffeomorphic to the product $S^{p_1} \times S^{p_2} \times, \ldots, \times S^{p_N}$.

CHAPTER V

BOCHNER FLAT KAEHLER MANIFOLDS WITH COMMUTING CURVATURE TENSOR AND RICCI OPERATOR

In the last chapter, we had shown that if a compact manifold admits a tight isometric immersion in R^m, then it is the product of spheres. In this chapter we have established similar kind of results in a different perspective and with different assumptions. Infact, we obtain a classification theorem for Bochner flat Kaehler manifolds under the additional condition that the Ricci operator commutes with the curvature tensor. We prove that under these conditions the underlying manifold is either a complex space form, locally Riemannian product of complex space forms or simply flat.

1. PRELIMINARIES:

Let (M,J,g) be a 2n-dimensional Kaehler manifold with Kaehler metric g and almost complex structure J. The Bochner curvature on a Kaehler manifold is considered as a complex version of Weyl conformal curvature tensor on a Riemannian manifold. A Kaehler metric is called a Bochner Kaehler metric if its Bochner curvature tensor [3] vanishes. An almost complex manifold with a Bochner Kaehler metric is called a Bochner flat Kaehler manifold.

The Bochner curvature tensor B of M is defined as follows:

$$(5.1.1) \quad B(X,Y) = R(X,Y) - \frac{1}{2n+4}[QX\wedge Y+X\wedge QY+QJX\wedge JY + JX\wedge QJY-2g(JX,QY)J-2g(JX,Y)QoJ] + \frac{r}{(2n+4)(2n+2)}[X\wedge Y+JX\wedge JY-2g(JX,Y)J]$$

for any tangent vectors X and Y, where R and Q are the Riemannian curvature tensor of M and Ricci operator which is a symmetric (1,1) tensor and r is the scaler curvature. $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Y,Z)X - g(X,Z)Y$.

In the case where the Kaehler manifold is Bochner flat, the curvature tensor is given by

$$(5.1.2) \quad R(X,Y) = \frac{1}{2n+4}[QX\wedge Y+X\wedge QY+QJX\wedge JY+JX\wedge QJY -2g(JX,QY)J -2g(JX,Y)QoJ] + \frac{r}{(2n+4)(2n+2)}[X\wedge Y+JX\wedge JY-2g(JX,Y)J]$$

There are following relations among g, J and Q.

$$(5.1.3) \quad J^2 = -I, \quad g(JX,Y) + g(X,JY) = 0$$

$$QoJ = JoQ, \quad g(QX,Y) = g(X,QY).$$

If $e_1, e_2, \ldots, e_{2n}$ is the orthonormal basis of $T_x(M)$, then

(5.1.4) $$Q(X,Y) = \sum_{i=1}^{2n} g(R(e_i,X)Y,e_i) = R(e_i,X;Y,e_i), \quad X,Y \in T_x(M)$$

For each plane section in the tangent space $T_x(M)$, the sectional curvature $K(p)$ for the plane p is defined by

(5.1.5) $$K(p) = R(X_1,X_2,X_2,X_1),$$

where X_1, X_2 is an orthonormal basis of p. If $K(p)$ is constant for all planes p in $T_x(M)$ & for all points $x \in M$, then M is called a space of constant curvature, For a space of constant curvature, we have

(5.1.6) $$R(X,Y)Z = C(g(Z,Y)X - g(Z,X)Y).$$

If p is invariant by the almost complex structure J, then $K(p)$ is called the holomorphic sectional curvature of p and is given by

(5.1.7) $$H(X) = R(X,JX,JX,X).$$

A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form and its curvature tensor is given by

$$(5.1.8) \qquad R(X,Y) = \rho/4\ [X \wedge Y + JX \wedge JY - 2g(JX,Y)J].$$

2. BOCHNER FLAT KAEHLER MANIFOLDS WITH COMMUTING CURVATURE TENSOR AND RICCI OPERATOR

We know that the Ricci operator and the curvature tensor commute in case of a complex space form. The purpose over here is to see the impact of the condition.

$$(5.2.1) \qquad R(X,Y) \circ Q = Q \circ R(X,Y)$$

On Bochner flat Kaehler manifolds. In order to prove the main results, we need the following key lemmas.

LEMMA (5.2.1): In a Bochner flat Kaehler manifold the following conditions are equivalent.

(i) $R(X,Y) \circ Q = Q \circ R(X,Y)$

(ii) $R(QX,X) = 0$

PROOF: We know that a skew symmetric operator commutes with a symmetric operator if and only if their product is skew symmetric. As $R(X,Y)$ is a skew symmetric endomorphism and the Ricci operator is symmetric. Therefore (i) holds if and only if $R(X,Y) \circ Q$ is skew symmetric that is if and only if

$$g(R(Y,Z)QX,X) = 0$$

or $$g(R(QX,X)Y,Z) = 0 \qquad \text{for all } Z,$$

i.e. $$R(QX,X) = 0$$

This proves the lemma.

In particular if a symmetric operator B commutes with the skew symmetric almost complex structure J, then BJ must be skew symmetric, i.e.

$$g(BJX,X) = 0$$

Hence we have the result.

<u>LEMMA</u> (5.2.2): For any symmetric operator B, commuting with almost complex structure J,

$$g(BJX,X) = 0. \tag{5.2.2}$$

Replacing Y by QX in Equation (5.1.2) and using lemma (5.2.1) and (5.2.2), we get

$$\begin{aligned} 0 = \frac{1}{(2n+4)} & [QX \wedge QX + X \wedge Q^2X + QJX \wedge JQX + JX \wedge Q^2JX \\ & -2g(JX,Q^2X)J - 2g(JX,QX)] \\ & + \frac{r}{(2n+4)(2n+2)} [X \wedge QX + JX \wedge JQX - 2g(JX,QX)J] \end{aligned}$$

on simplification, we get

$$\frac{1}{(2n+4)} [X \wedge Q^2X + JX \wedge JQ^2X] - \frac{r}{(2n+4)(2n+2)} [X \wedge QX + JX \wedge JQX] = 0$$

or $[X \wedge Q^2X + JX \wedge JQ^2X] - \frac{r}{2(n+1)} [X \wedge QX + JX \wedge JQX] = 0$

i.e.

(5.2.3) $$X \wedge BX + JX \wedge JBX = 0$$

where

(5.2.4) $$B = Q^2 - \frac{r}{2(n+1)} Q.$$

Based on this and lemma (5.2.2), we establish the following lemma.

LEMMA (5.2.3): Let M be a Bochner flat Kaehler manifold of dimension $2n$, satisfying

$$R(X,Y) \circ Q = Q \circ R(X,Y), \text{ then}$$

(5.2.5) $$Q^2 - \frac{r}{2(n+1)} Q - \rho I = 0$$

where ρ is a C^∞-real valued function on M.

PROOF: Let us compute the inner product of $X \wedge BX$ and $JX \wedge JBX$. Using lemma (5.2.2), we get,

(5.2.6) $$g(X \wedge BX(Z), JX \wedge JBX(Z))$$

$$= g(X,Z)g(BJX,Z)g(JBX,X)-g(BX,Z)g(JX,Z)g(X,BJX) = 0$$

From Equation (5.2.3) and (5.2.6), it follows that

$$X \wedge BX = 0$$

Therefore, there exists a C^∞-real valued function ρ on M satisfying

$$BX = \rho X$$

Substituting in Equation (5.2.4), we get the result.

In respect of the eigen vectors of Q, we note that

LEMMA (5.2.4): If X is the eigen vector of Q, then so is JX with the same eigen value.

The Proof follows from the fact that Q commutes with J.

Now we are in a position to establish the main result of this chapter.

THEOREM (5.2.1): Let (M,J,g) be a connected Kaehler manifold of dimension $2n$ with vanishing Bochner curvature tensor and satisfying the condition

$$R(X,Y) \circ Q = Q \circ R(X,Y)$$

Where R is the curvature tensor of M and Q is the Ricci tensor. Then M is one of the following:

i) Flat

ii) A complex space form

iii) Locally Riemannian product of two space forms of constant curvature C and $-C$ respectively.

PROOF: Since Q satisfies the second degree polynomial (5.2.5) of lemma (5.2.3), it follows that at every point $x \in M$, Q has two eigen values say λ and μ viz

$$\frac{r \pm \sqrt{r^2+4(n-1)\rho}}{2(n-1)}$$

Let W be the set of points $x \varepsilon M$ on which Q has distinct eigen values $\lambda(x)$ and $\mu(x)$.

i.e. $\qquad r^2 + 4(n-1)\rho \neq 0$ on W.

Clearly W is an open subset of M. We have

$$\lambda + \mu = \frac{r}{2(n+1)}$$

and $\qquad \lambda \mu = -\rho$

Lemma (5.2.4) shows that the multiplicities of λ and μ are even. The eigen values $\lambda(x)$ and $\mu(x)$ are defined and continuous on all of M and distinct on W. Let W_o be the connected component of W containing x_o. Define two distributions on W_o as follows

$$D_1(x) = \{ X \ \varepsilon \, T_x(M) \ / \ QX = \lambda(x)\, X \}$$

$$D_2(x) = \{ X \ \varepsilon \, T_x(M) \ / \ QX = \mu(x)\, X \}.$$

It is seen that $D_1(x)$ and $D_2(x)$ are:

(i) J-invariant, as JX is also an eigen vector of Q with the same eigen value as X (Lemma (5.2.4))

(ii) Involutive, as $QX = \lambda(x)X$ and $QY = \lambda(x)Y$ implies $Q[X,Y] = \lambda(x)[X,Y]$.

(iii) Parallel, i.e. $X,Y \in D_i(x)$ implies $\nabla_X Y \in D_i(x)$ (See [34]).

The sectional curvature $K(X,Y)$ and the holomorphic sectional curvature $K(X,JX)$ of M are given by

$$H(X) = C\,, \quad \text{if } X \in D_1$$

$$K(X,Y) = 0\,, \quad \text{if } X \in D_1,\ Y \in D_2$$

$$H(X) = -C\,, \quad \text{if } X \in D_2$$

where $C = \frac{2(\lambda-\mu)}{(n+2)}$

Therefore the leaves of D_1 and D_2 are totally geodesic submanifold of W_o. But the connectedness of M and continuity of λ and μ implies that $W_o = M$, and hence the leaves of D_1 and D_2 are totally geodesic submanifold of M. Moreover under these conditions λ and μ can be shown to be constants (see [34]). Now we discuss the following cases:

<u>Case 1</u>: <u>If $\lambda = \mu$</u>.

We have $\lambda + \mu = \frac{r}{2(n+1)}$

that is $r = 4\lambda(n+1)$.

Now from Equation (5.1.2), we have

$$R(X,Y) = \frac{1}{2n+4}\,[QX\wedge Y + X\wedge QY + QJX\wedge JY + JX\wedge QJY$$

$$-2g(JX,Y)Q \circ J - 2g(JX,QY)J]$$

$$-\frac{2}{(2n+4)}\,[X\wedge Y + JX\wedge JY - 2g(JX,Y)J] = 0$$

This means M is flat. Hence part (i) of the theorem follows:

Case 2: If $\lambda \neq \mu$ and one of the λ and μ is zero.

Suppose $\mu = 0$

Hence from $\lambda + \mu = \frac{r}{2(n+1)}$, we have

$$r = 2\lambda\,(n+1).$$

Now by Equation (5.1.2), we have

$$R(X,Y) = \frac{\lambda}{2n+4}\,[X\wedge Y + JX\wedge JY - 2g(JX,Y)J].$$

Which shows that M is a complex space form. Hence part (ii) of the theorem follows:

Case 3: Both λ and μ are non-zero and $\lambda \neq \mu$.

In this case from (5.2.5) part (iii) of the theorem follows i.e. M is locally product manifold of complex space forms of constant holomorphic sectional curvature C and -C.

Q.E.D.

Examples: Einstein Kaehler manifold and the complex space form can be seen to be satisfying the hypothysis of the theorem.

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